# Announcements

1) HW #3 deadline extended to Friday

**Definition:** (finite, middle)

Let $f$ be a function that is continuous on $[a,b]$ except for a point $c$ with $a < c < b$.

Then $\int_a^b f(x)\,dx$ is definid as

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dy$$

provided both improper integrals exist

Let's try $\int_{-2}^{1} \frac{1}{x^2}\,dx$ again!

$$\int_{-2}^{1} \frac{1}{x^2}\,dx = \int_{-2}^{0} \frac{1}{x^2}\,dx + \int_{0}^{1} \frac{1}{x^2}\,dx$$

$$= \lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^2}\,dx + \lim_{s \to 0^+} \int_{s}^{1} \frac{1}{x^2}\,dx$$

Since $\frac{1}{x^2}$ is discontinuous at $x = 0$.

$$\lim_{t \to 0^-} \int_{-2}^{t} \frac{1}{x^2}\,dx = \lim_{t \to 0^-} \int_{-2}^{t} x^{-2}\,dx$$

$$= \lim_{t \to 0^-} \left. \frac{x^{-1}}{-1} \right|_{-2}^{t}$$

$$= \lim_{t \to 0^-} \left( \frac{t^{-1}}{-1} - \left( \frac{(-2)^{-1}}{-1} \right) \right)$$

$$= \lim_{t \to 0^-} \left( -\frac{1}{t} - \frac{1}{2} \right)$$

$= \infty$, so this

integral does not exist.

This shows that

$$\int_{-2}^{1} \frac{1}{x^2}\,dx \text{ does not exist.}$$

Note: In order for the integral to exist, both pieces have to exist. In order for it not to exist, one of the pieces must not exist

If the integral exists, we say it is Convergent. If not, we say it is divergent

# Infinite Domains

**Definition:** (1-sided) Let $f$ be continuous on the interval $[a,\infty)$ for some number $a$

We define

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx}$$

provided the limit exists.

Similarly, if $f$ is continuous on $(-\infty, a]$, we define

$$\int_{-\infty}^{a} f(x)\,dx = \lim_{s \to -\infty} \int_{s}^{a} f(x)\,dx$$

provided the limit exists.

Example 2:

$$\int_{e^2}^{\infty} \frac{1}{\ln(x^x)}\, dx$$

Write as $\lim_{t\to\infty} \int_{e^2}^{t} \frac{1}{\ln(x^x)}\, dx$

$$= \lim_{t\to\infty} \int_{e^2}^{t} \frac{1}{x\ln(x)}\, dx$$

$u = \ln(x)$ $\quad u(e^2) = 2$

$du = \frac{1}{x}\, dx$ $\quad u(t) = \ln(t)$

$$\lim_{t \to \infty} \int_2^{\ln(t)} \frac{1}{u}\, du$$

$$= \lim_{t \to \infty} \ln(u) \Big|_2^{\ln(t)}$$

$$= \lim_{t \to \infty} \Big( \ln(\ln(t)) - \ln(2) \Big)$$

$$= \lim_{t \to \infty} \ln\left( \frac{\ln(t)}{2} \right) \quad \text{(log rules)}$$

$\lim_{t \to \infty} \ln(t) = \infty$, so

$$\lim_{t \to \infty} \ln\left( \frac{\ln(t)}{2} \right) = \infty$$

So the integral does not exist.

**P-rule:** Let $a > 0$ and consider

$$\int_a^\infty \frac{1}{x^p}$$ for a real number $p$.

The integral exists if $p > 1$.

The integral does not exist if $p \leq 1$

You are now free to quote this rule!

Example 3:

$$\int_{-2}^{\infty} x e^{-x}\, dx$$

$$= \lim_{t \to \infty} \int_{-2}^{t} x e^{-x}\, dx$$

integrate by parts

$u = x$ $\qquad v = -e^{-x}$

$du = dx$ $\qquad dv = e^{-x}\, dx$

$$\int_{-2}^{t} x e^{-x}\, dx = -x e^{-x} \Big|_{-2}^{t} + \int_{-2}^{t} e^{-x}\, dx$$

$$\int_{-2}^{t} xe^{-x}\,dx = -xe^{-x}\Big|_{-2}^{t} + \int_{-2}^{t} e^{-x}\,dx$$

$$= \frac{-x}{e^{x}}\Big|_{-2}^{t} + \int_{-2}^{t} e^{-x}\,dx$$

$$= \frac{-x}{e^{x}}\Big|_{-2}^{t} - e^{-x}\Big|_{-2}^{t}$$

$$= \left(\frac{-x-1}{e^{x}}\right)\Big|_{-2}^{t}$$

Now take limit as $t \to \infty$!

$$\lim_{t\to\infty}\left(\frac{-x-1}{e^{x}}\right)\Bigg|_{-2}^{t}$$

$$=\lim_{t\to\infty}\left(\frac{-t-1}{e^{t}}+e^{2}\right)$$

$$=\left(\lim_{t\to\infty}\frac{-t-1}{e^{t}}\right)+e^{2}$$

$$\lim_{t\to\infty}\frac{-t-1}{e^{t}}\overset{\text{l'H}}{=}\lim_{t\to\infty}\frac{-1}{e^{t}}$$

$$=0$$

Then $\int_{-2}^{\infty} x e^{-x}\, dx = e^{2}$,

so the integral exists.

**Definition:** (2-sided) Suppose $f$ is continuous everywhere.

Define

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \int_{a}^{\infty} f(x)\,dx + \int_{-\infty}^{a} f(x)\,dx}$$

where $a$ is any real number (your choice) and provided both integrals exist

If the integral exists,
we say it is convergent
If not, we say it is
divergent.

**Comparison** Suppose $0 \le f(x) \le g(x)$ on the interval $[a, \infty)$ and both $f$ and $g$ are continuous

Then (always)

1) If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then

$\displaystyle\int_a^\infty f(x)\,dx$ converges

2) If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then

$\displaystyle\int_a^\infty g(x)\,dx$ diverges.

**Philosophy:** only highest powers matter

**Example 4.** $$\int_3^\infty \frac{x^3-3x^2+1}{x^5+2}\,dx$$

Since by our philosophy, only the highest powers matter, this is "like"

$$\int_3^\infty \frac{x^3}{x^5}\,dx = \int_3^\infty \frac{1}{x^2}\,dx$$

so this integral exists by p-rule ($p=2>1$)

So the original integral should exist! Means convergent.

Find a function that is bigger and whose integral converges.

$$f(x) = \frac{x^3 - 3x^2 + 1}{x^5 + 2}$$

$$\leq \frac{x^3 - 3x^2 + 1}{x^5}$$ since $x^5 < x^5 + 2$

$$\leq \frac{x^3 + 1}{x^5}$$ since $x^3 - 3x^2 + 1 \leq x^3 + 1$

Know $f(x) \le \dfrac{x^3+1}{x^5}$

$$\le \frac{x^3+x^3}{x^5} \quad \text{since } 1 \le x^3$$

$$= \frac{2x^3}{x^5} \quad \text{since } x \ge 3$$

$$= \frac{2}{x^2}$$

Now use p-rule to conclude

$$\int_3^\infty \frac{x^3-3x^2+1}{x^5+2}\,dx \text{ converges}$$

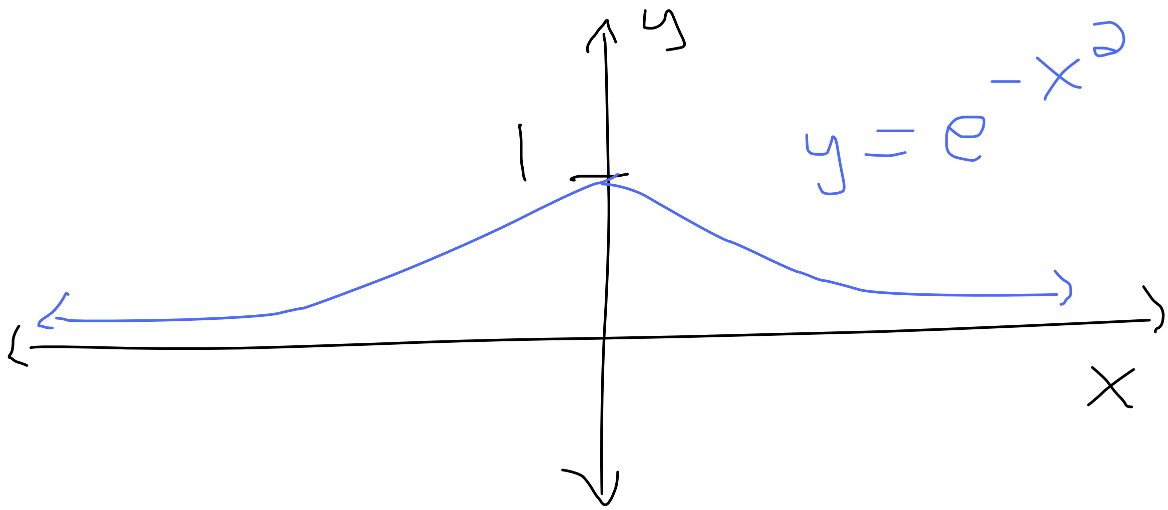
by comparison.

Example 5:

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

The enemy!

graph of $e^{-x^2}$




this is (a version of) the bell curve!

In Calc 3, you will learn how to compute this integral (I think it's $\sqrt{\pi}$)

Let's show the integral is convergent using comparison.

Since $e^{-x^2}$ is an even function (even: $f(-x)=f(x)$),

$$\int_0^t e^{-x^2}\,dx = \int_{-t}^0 e^{-x^2}\,dx,$$ so

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx.$$

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{some number, } \leq 1.} + \int_1^\infty e^{-x^2}dx$$

All we care about now is

$$\int_1^\infty e^{-x^2}dx$$

When $x \geq 1$, $e^{-x^2} \leq e^{-x}$.

Same as $\frac{1}{e^{x^2}} \leq \frac{1}{e^x}$

Same as $e^{x^2} \geq e^{x}$

Is this true? Take ln of both sides

$$\ln(e^{x^2}) \overset{?}{\geq} \ln(e^{x})$$

$$x^2 \geq x \quad \text{when } x \geq 1.$$

So $e^{-x^2} \leq e^{-x}$ when $x \geq 1$.

By comparison,

$$\int_1^{\infty} e^{-x^2}\,dx \leq \int_1^{\infty} e^{-x}\,dx$$

$$\int_1^{\infty} e^{-x}\,dx = \lim_{t\to\infty} \int_1^{t} e^{-x}\,dx$$

$$= \lim_{t\to\infty} -e^{-x} \Big|_1^{t}$$

$$= \lim_{t\to\infty} \left( -\frac{1}{e^t} + \frac{1}{e} \right)$$

$$\lim_{t\to\infty} \frac{1}{e^t} = 0 \text{ , so}$$

$$\int_1^{\infty} e^{-x}\,dx = \frac{1}{e}$$

So:
$$\int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$= 2\int_0^{\infty} e^{-x^2}\,dx$$

$$= 2\left(\int_0^1 e^{-x^2}\,dx + \int_1^{\infty} e^{-x^2}\,dx\right)$$

$$\leq 2\left(1 + \frac{1}{e}\right) < \infty,$$

which says our integral converges!

# Sequences and Series

Chapter 11 Not like english!

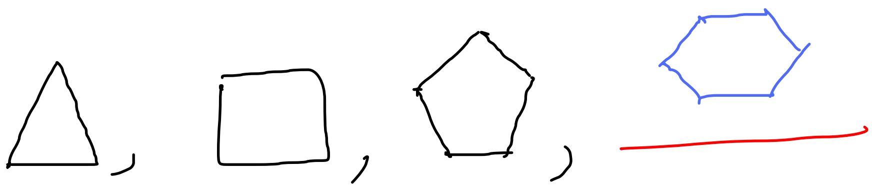
## Sequences (11.1)

A sequence is an infinite list of objects, indexed by the counting numbers

$$(a_1, a_2, a_3, \ldots)$$

1st object 2nd object 3rd object

## Examples: Find the pattern!


, 7-sided, etc

O, T, T, F, F, S, S, E, N, T

one two three four five six seven eight nine ten

1, 11, 21, 1211, 111221, 312211,

13112221

We will be interested in sequences of numbers.

Can just think of a sequence as a function from the counting numbers to the real numbers

$$a_1 = f(1)$$
$$a_2 = f(2)$$
$$\vdots$$
$$a_n = f(n)$$

## Example 6: (Fibonacci)

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ___

the next term is the sum of the previous 2

Recursive procedure

$$a_n = a_{n-1} + a_{n-2}$$

$(a_4 = a_3 + a_2,\ a_7 = a_6 + a_5, \text{etc})$

## Example 7: (find $a_n$)

$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \_\_\_$

$$a_n = \frac{1}{n}$$

$1, 5, 25, 125, 625, \_\_\_$

$$a_n = 5^{n-1}$$

$5, 7, 9, 11, 13, \_\_\_$

$$a_n = 2n+3$$